Integral po proizvoljnom (otvorenom) skupu

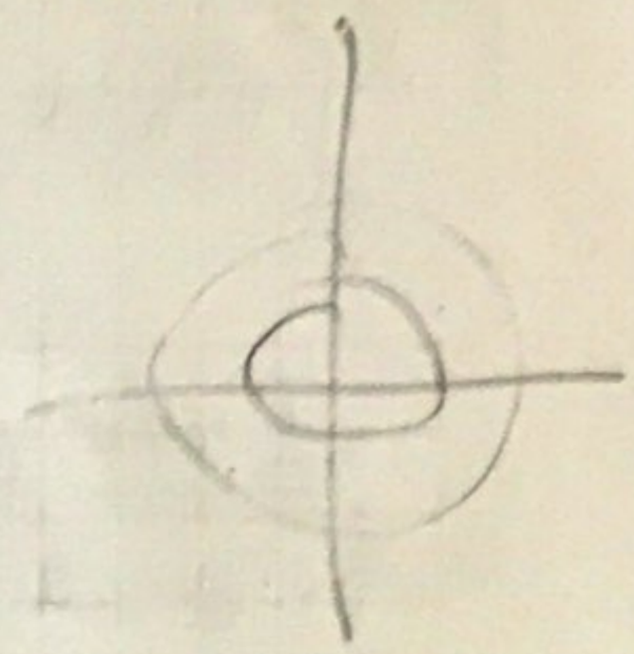
Def $E \subseteq \mathbb{R}^n$, $\{E_n\}_{n=1}^{\infty} \in J$, td. $E = \bigcup_{n=1}^{\infty} E_n$ i $\overline{E_n} \subseteq E_{n+1}$ tada je $\{E_n\}_{n=1}^{\infty}$ monotoni pokrivač za E.

Def Neka je $f: E \to \mathbb{R}$. Ako za svaki monotoni pokrivač $\{E_n\}_{n=1}^{\infty}$ skupa E postoji $\lim\limits_{n\to\infty} \int_{E_n} f < \infty$ i ako ne zavisi od izbora $\{E_n\}_{n=1}^{\infty}$ tada kažemo da $\int_E f$ konvergira, tj $\int_E f = \lim\limits_{n\to\infty} \int_{E_n} f$

Teorema Ako je $f \geq 0$, $f: E \to \mathbb{R}$ i postoji bar jedan monotoni pokrivač $\{E_n\}_{n=1}^{\infty}$ skupa E i $\exists \lim\limits_{n\to\infty} \int_{E_n} f$ tada $\int_E f$ konvergira.

Teorema $\int_E f$ konvergira $\Leftrightarrow \int_E |f|$ konvergira

(1) $\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$



R: $e^{-x^2-y^2} > 0 \quad \forall (x,y) \in \mathbb{R}^2$

- f-ja poz, nije bitno kakav ćemo pokrivač izabrati

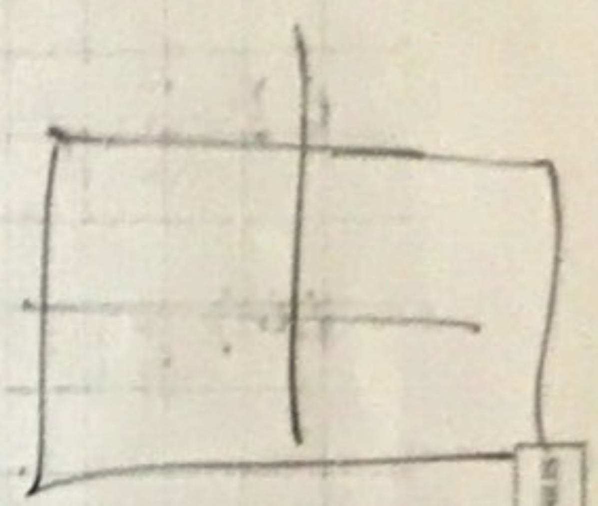
neka je $K_n = \{(x,y) \mid x^2+y^2 \le n^2\}$

$\bigcup K_n = \mathbb{R}^2$, K_n - monoton pokrivač

$$\iint\limits_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \iint\limits_{K_n} e^{-x^2-y^2}\,dx\,dy = \text{(uvedemo polarne koo.)}$$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_0^n e^{-r^2}\, r\,dr\,d\varphi = \lim_{n\to\infty} \pi \int_0^n 2r e^{-r^2}\,dr =$$

$$= \lim_{n\to\infty} \pi\,(1-e^{-n^2}) = \pi$$



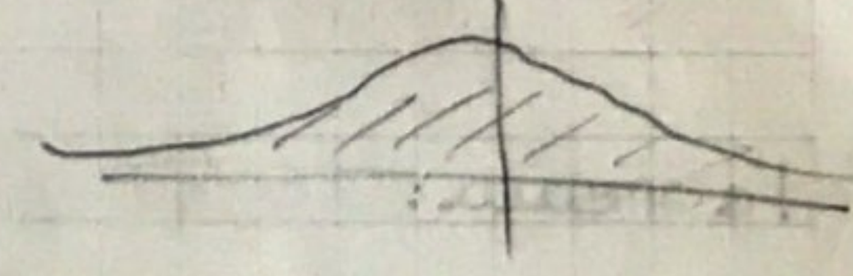
$I_n = [-n,n]\times[-n,n]$ - hoćemo da pokrijemo sa kvadratima

$$\pi = \lim_{n\to\infty} \iint\limits_{I_n} e^{-x^2-y^2}\,dx\,dy = \lim_{n\to\infty} \int_{-n}^{n}\int_{-n}^{n} e^{-x^2}\cdot e^{-y^2}\,dx\,dy =$$

tj. što znamo

$$= \lim_{n\to\infty} \int_{-n}^{n} e^{-y^2}\,dy \cdot \int_{-n}^{n} e^{-x^2}\,dx = \lim_{n\to\infty} \left(\int_{-n}^{n} e^{-x^2}\,dx\right)^2$$

$$\Rightarrow \sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2}\,dx \quad \text{Ojlerov integral} \iff \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$



(2) U zavisnosti od p ispitati kv. int:

$$\iint\limits_{\mathbb{R}^2} \frac{dx\,dy}{(1+x^2+y^2)^p}$$

R: $f(x,y) = \dfrac{1}{(1+x^2+y^2)^p} > 0$

Opet izaberimo $K_n = \{(x,y) \mid x^2+y^2 \le n^2\}$ - koncentrične krugove

$$I = \lim_{n\to\infty} \int_0^{2\pi}\int_0^n \frac{r\,dr\,d\varphi}{(1+r^2)^p} = \left(\begin{matrix} 1+r^2 = s \\ 2r\,dr = ds \end{matrix}\right) = \lim_{n\to\infty} \pi \int_1^{1+n^2} \frac{ds}{s^p}$$

za $p = 1 \Rightarrow \pi \lim\limits_{n\to\infty} \ln(1+n^2)$ → Da bi postojao limes t.j. da bi int. kv.

$p \ne 1 \Rightarrow \pi \lim\limits_{n\to\infty} \left.\dfrac{s^{1-p}}{1-p}\right|_1^{1+n^2}$ $\quad 1-p<0 \Rightarrow p>1$

$\Rightarrow p>1$ kv.

$p \le 1$ div.

$\ulcorner$

$$\iint_{\mathbb{R}^2} e^{-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$$

$$E_n = \left\{ (x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2} \le n^2 \right\}$$

$\lrcorner$

③ $$\iint_{\mathbb{R}^2} e^{-x^2-2x-1-y^2-y}\,dx\,dy =$$

R: $\ulcorner$ $-(x^2+2x+1+y^2+y) = -\left((x+1)^2+(y+\frac{1}{2})^2-\frac{1}{4}\right)$ $\lrcorner$

smjena: $u = x+1$

$v = y+\frac{1}{2}$

$$= e^{\frac{1}{4}}\iint_{\mathbb{R}^2} e^{-(x+1)^2-(y+\frac{1}{2})^2}\,dx\,dy \overset{(*)}{=} e^{\frac{1}{4}}\iint_{\mathbb{R}^2} e^{-u^2-v^2}\,du\,dv =$$

Jakobijan $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$$= \pi \cdot e^{\frac{1}{4}}$$

II način

(*)

$$E_n = \left\{ (x,y) \mid (x+1)^2+(y+\frac{1}{2})^2 \le n^2 \right\}$$

$$\bigcup E_n = \mathbb{R}^2$$

$\ulcorner$ f-ja je pozitivna pa nije bitno koji ćemo pokrivač uzeti $\lrcorner$

$$(*) = e^{\frac{1}{4}} \lim_{n\to\infty} \iint_{E_n} e^{-(x+1)^2-(y+\frac{1}{2})^2}\,dx\,dy = \ldots$$

polarne koordinate

④ Dokazati da integral $\iint_{\mathbb{R}^2} \sin(x^2+y^2)\,dx\,dy$ divergira.

R: $I_n = [-n,n]^2$

$$\lim_{n\to\infty} \iint_{I_n} \sin(x^2+y^2)\,dx\,dy =$$

$$\sin(x^2+y^2) = \sin(x^2)\cos(y^2)+\cos(x^2)\sin(y^2)$$

$$= \lim_{n\to\infty} \int_{-n}^{n}\int_{-n}^{n} (\sin(x^2)\cos y^2 + \cos x^2 \sin y^2)\,dx\,dy =$$

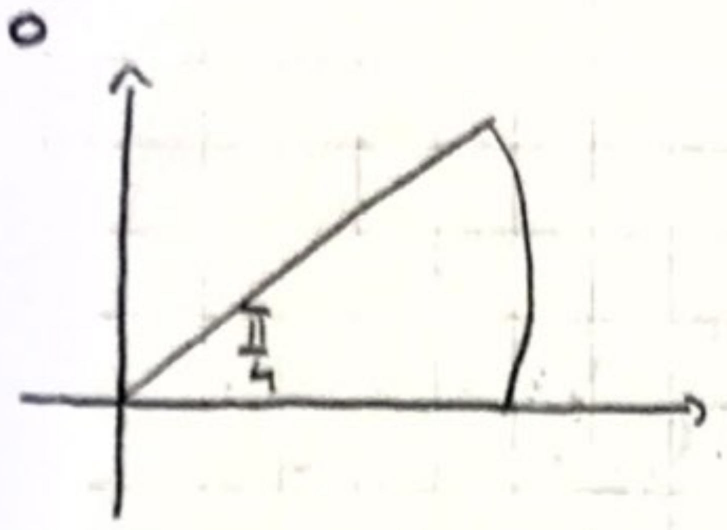
$$= \lim_{n\to\infty} \left(\int_{-n}^{n} \cos y^2\,dy \int_{-n}^{n} \sin x^2\,dx + \int_{-n}^{n} \cos(x^2)\,dx \int_{-n}^{n} \sin y^2\,dy \right) =$$

$$= 2 \lim_{n\to\infty} \left(\int_{-n}^{n} \cos x^2 \, dx \int_{-n}^{n} \sin x^2 \, dx \right) =$$

$\sqrt{\ }$

$$\int_0^\infty \sin(x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Fresnelovi integrali



$e^{iz^2} \ldots$

$$\int_{-\infty}^{+\infty} \sin(x^2)\, dx = \int_{-\infty}^{+\infty} \cos(x^2)\, dx = \sqrt{\frac{\pi}{2}}$$

$\Downarrow$

$$= 2 \cdot \sqrt{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}} = \pi \qquad (*)$$

$$K_n = \{ (x,y) \mid x^2 + y^2 \le 2n\pi \}$$

$$\lim_{n\to\infty} \iint_{K_n} \sin(x^2+y^2)\, dx\, dy = \lim_{n\to\infty} \int_0^{2\pi} \int_0^{\sqrt{2n\pi}} \sin(r^2)\, r\, dr\, d\varphi =$$

$$= \pi \lim_{n\to\infty} \int_0^{\sqrt{2n\pi}} \sin(r^2)\, 2r\, dr = \left(\begin{matrix} t = r^2 \\ dt = 2r dr \end{matrix} \right) = \pi \lim_{n\to\infty} \int_0^{2n\pi} \sin t \, dt =$$

$$= \pi (1 - \underbrace{\cos(2n\pi)}_{1}) = 0 \qquad (**)$$

iz $(*)$ i $(**)$ $\Rightarrow$ integral ne konvergira

$$K_n' = \{ (x,y) \mid x^2 + y^2 \le 2n\pi + \frac{\pi}{2} \}$$

$$\lim_{n\to\infty} \iint_{K_n'} \sin(x^2+y^2)\, dx\, dy$$

$$= \pi \left(1 - \cos\left(2n\pi + \frac{\pi}{2}\right)\right) = \pi$$

(5.) $$\iiint_{\mathbb{R}^3 \setminus \{(0,0,0)\}} \frac{e^{-x^2-y^2-z^2}}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz$$

$f \ge 0$

Γ $\quad \frac{e^{-t}}{\sqrt{t}} \xrightarrow{t\to\infty} 0 \quad , t \ne 0$

$$I = \underbrace{\iiint_{0 < x^2+y^2+z^2 \le 1} f}_{I_1} + \underbrace{\iiint_{1 \le x^2+y^2+z^2 \le n^2} f}_{I_2} \quad , f \ge 0$$

$\underline{\underline{I_1}}$

$D_n = \left\{ (x,y,z) \;\middle|\; \frac{1}{n^2} \leq x^2+y^2+z^2 \leq 1 \right\}$ - prsten u $\mathbb{R}^3$

$\overline{D_n} \subseteq D_{n+1}$

$\cup D_n = \left\{ 0 < x^2+y^2+z^2 \leq 1 \right\}$

$$\lim_{n\to\infty} \iiint_{D_n} f\,dx\,dy\,dz = \left(\begin{array}{ll} x = r\cos\varphi\sin\theta & \varphi \in [0, 2\pi] \\ y = r\sin\varphi\sin\theta & \theta \in [0, \pi] \\ z = r\cos\theta & r \in (\frac{1}{n}, 1) \end{array} \right) =$$

$|J| = r^2 \sin\theta$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_0^{\pi}\int_{\frac{1}{n}}^{1} \frac{e^{-r^2}}{\cancel{r}}\, r^{\cancel{2}} \sin\theta \, dr\,d\theta\,d\varphi =$$

$$= 2\pi \lim_{n\to\infty} \int_0^{\pi} \sin\theta \int_{\frac{1}{n}}^{1} e^{-r^2}\, r\,dr =$$

$$= 2\pi \underbrace{\lim_{n\to\infty} \int_{\frac{1}{n}}^{1} e^{-r^2}\, 2r\,dr}_{} = 2\pi \lim_{n\to\infty} \left(e^{-\frac{1}{n^2}} - e^{-1} \right) = 2\pi\,(1 - e^{-1})$$

$\underline{\underline{I_2}}$

$r \in (1, n)$

$$\ldots = 2\pi \lim_{n\to\infty} (e^{-1} - e^{-n^2}) = 2\pi e^{-1}$$

$$\Rightarrow I = I_1 + I_2 = 2\pi$$

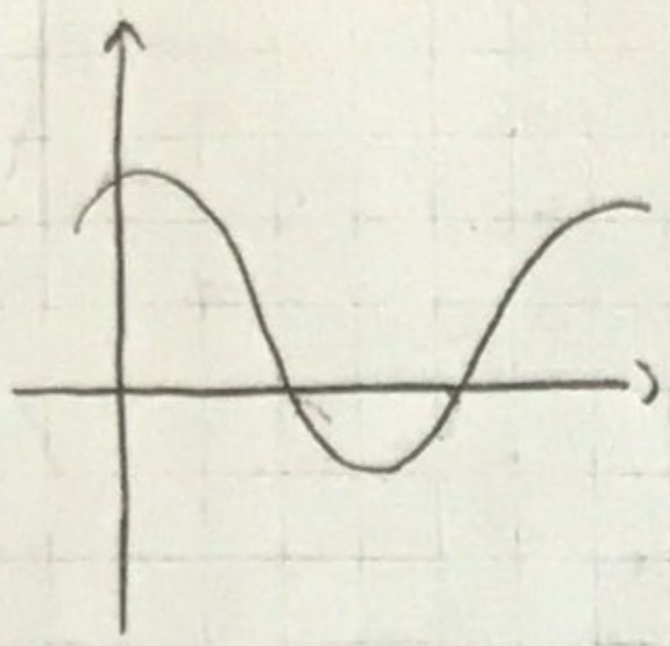
⌜Moglo je i sve odjednom, da se uzme interval $(\frac{1}{n}, n)$⌟

Teorema: $\int_A f < \infty \iff \int_A |f| < \infty$

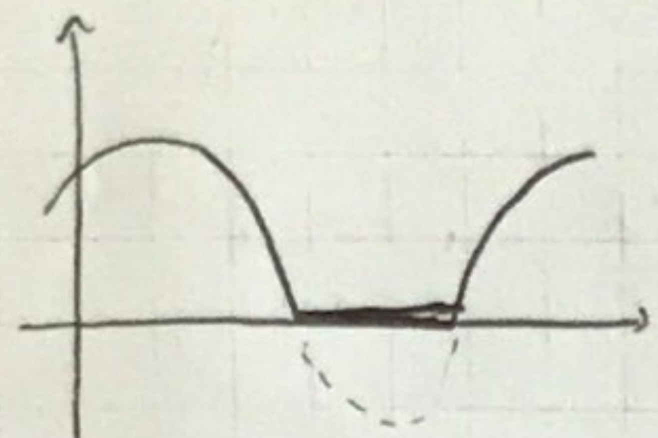
$f^+ = \max\{f, 0\}$

$f^- = -\min\{f, 0\} = \max\{-f, 0\}$

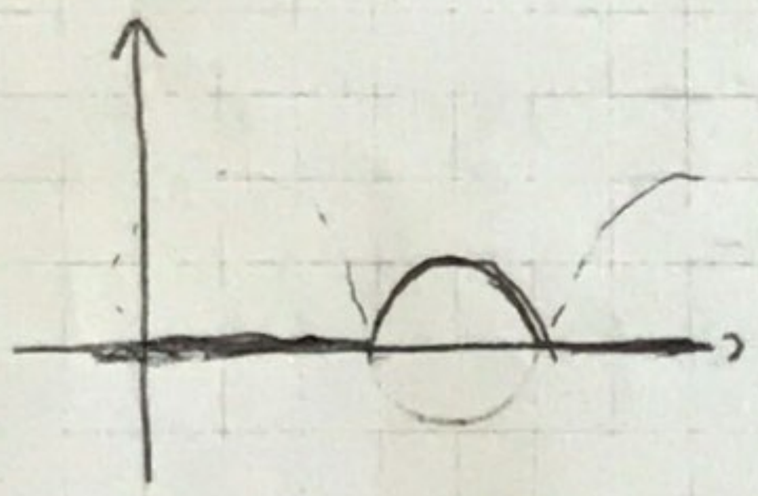
pr.



f



f^+



f^-

$$\boxed{f = f^+ - f^-}$$

$$\boxed{|f| = f^+ + f^-}$$

6. Pokazati da integral

$$\iint\limits_{\substack{x \geq 1 \\ y \geq 1}} \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx\, dy$$

divergira iako postoje ponovljeni integrali

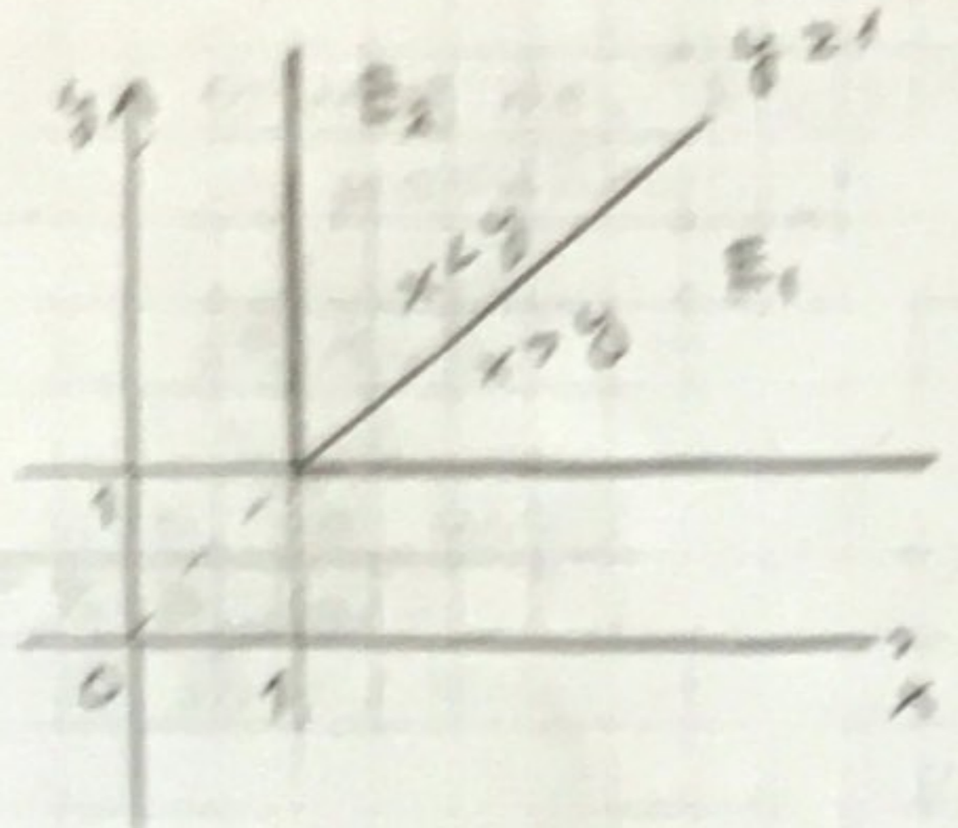
$$\int_1^{+\infty} dx \int_1^{+\infty} \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy = -\frac{\pi}{4}$$

$$\int_1^{+\infty} dy \int_1^{+\infty} -\!\!\shortparallel\!\!-\, dx = \frac{\pi}{4}$$

$$\left(\frac{y}{x^2+y^2}\right)'_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

↪ prim. f-ja

Naš int. div. ako $\iint\limits_{\substack{x\ge1\\ y\ge1}} |f| \, dx$, pa posmatrajmo $I = \iint \frac{|x^2-y^2|}{(x^2+y^2)^2}\,dxdy$



$$|x^2-y^2| = \begin{cases} x^2-y^2, & x^2>y^2 \text{ t.j. } x>y \\ y^2-x^2, & x^2<y^2 \text{ t.j. } x<y \end{cases}$$

Pravimo oblast $K_n = \{(x,y) \mid x\in[1,n],\ y\in[1,n]\}$

int. po K_n

$$I_n = \int_1^n\int_1^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx + \int_1^n\int_x^n \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx >$$

$$> \int_1^n\int_1^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \int_1^n \left(\frac{y}{x^2+y^2}\right)\Big|_1^x dx =$$

$$I_n = \int_{K_n} |f|$$

$$= \int_1^n \frac{x}{2x^2}\,dx - \int_1^n \frac{1}{x^2+1}\,dx = \frac12\int_1^n \frac1x\,dx - \int_1^n \frac{1}{x^2+1}\,dx =$$

$$= \frac12 \ln x\Big|_1^n - \operatorname{arctg} x\Big|_1^n = \frac12(\ln n - \cancel{\ln 1}^{\,0}) - (\operatorname{arctg} n - \operatorname{arctg} 1) =$$

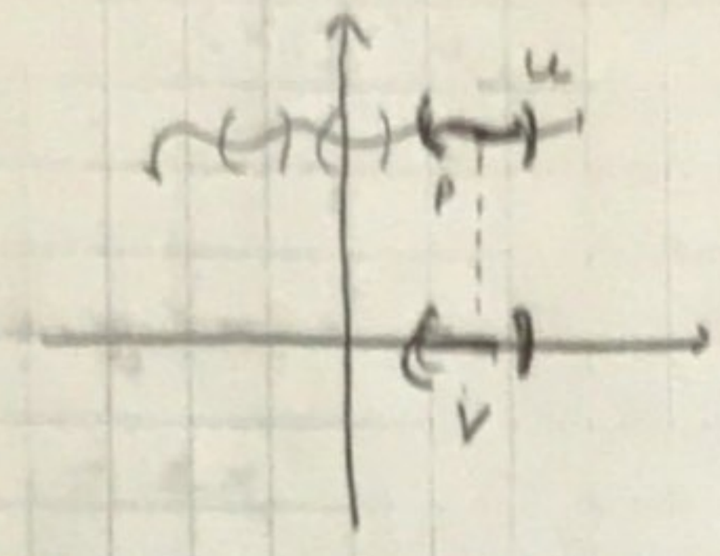
$$= \frac12 \ln n - \operatorname{arctg} n + \frac{\pi}{4} \xrightarrow{n\to+\infty} +\infty$$

$\Rightarrow$ naš integral divergira

V 23.04. 2018. (XI nedelja)

Mnogostrukosti

Def. Za skup $M^n \subseteq \mathbb{R}^N$ (topološki prostor) $n \le N$ kažemo da je n-dim mnogostrukost ako $\forall p \in M^n$ postoji okolina $U \ni p$, $U \subseteq M^n$ i postoji okolina $V \subseteq \mathbb{R}^n$ i $\exists$ $h: U \to V$ (homeomorfizam).

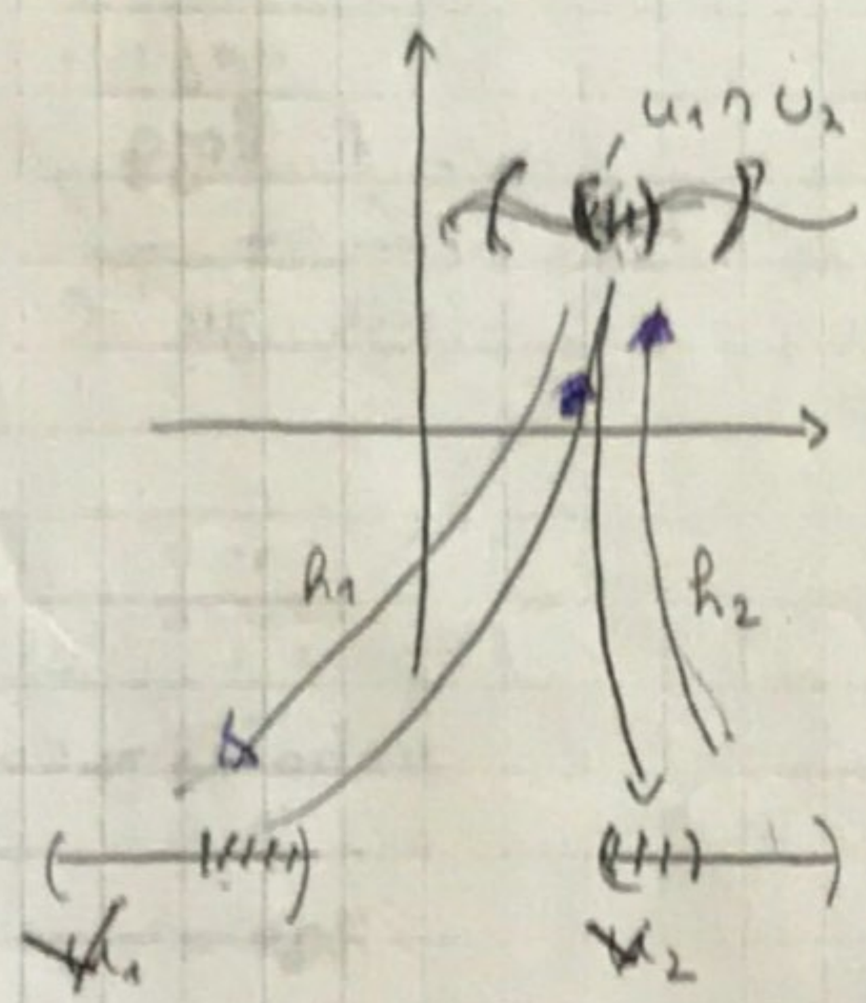


- kriva je mnogostrukosti 1 u prostoru $\mathbb{R}^2$

(U, h) - karta mnogostrukosti

$\{(U_i, h_i)\}$ - atlas mnogostrukosti (skup svih karti date mnogostrukosti)

$\bigcup_{i \in I} U_i = M^n$



$h_{12} = h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \to h_2(U_1 \cap U_2)$

↳ difeomorfizam

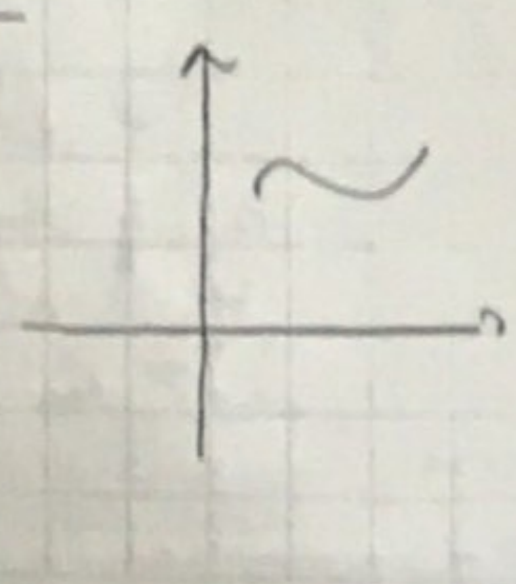
Def. Za n-dim mnogostrukost M^n kažemo da je glatka klase C^k ako $(\forall p \in M^n)(\exists (U, h), p \in U)$ $h: U \to V$, $\underline{g = h^{-1} : V \to U}$ (n dim, N dim) zadovoljava sledeće uslove: 1) $g \in C^k$,

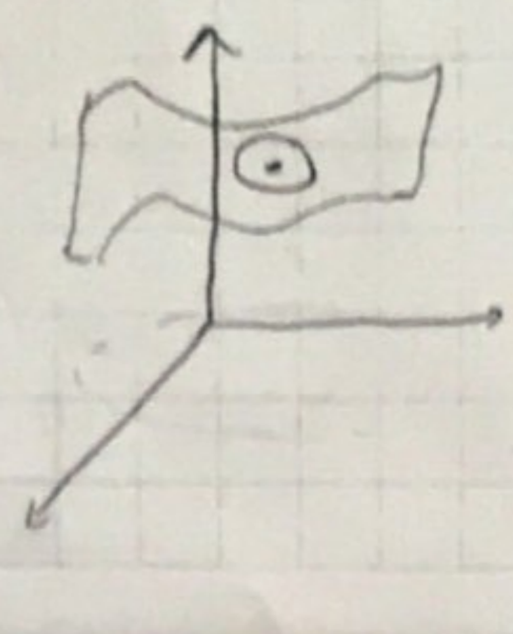
2) $\text{rang}(g') = n$

primjer Grafik glatke f-je $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ je glatka mnogostrukost

$\Gamma_f = \{(x_1, x_2, \dots, x_n, f(x_1, \dots, x_n)) \mid (x_1, \dots, x_n) \in U\} \subseteq \mathbb{R}^{n+1}$

Γ





$h : \Gamma_f \to V$ $(\subseteq \mathbb{R}^n)$ projekcija na prve n koordinate

$\forall p \in \Gamma_f$ $p = (x_0^1, x_0^2, \dots, x_0^n, f(x_0^1, \dots, x_0^n))$

$h : \Gamma_f \to U \subseteq \mathbb{R}^n$

$(x_1, \dots, x_n, f(x_1, \dots, x_n)) \xrightarrow{h} (x_1, \dots, x_n)$, h - homeomorfizam

Da bismo pokazali da je glatka mnogostrukost, moramo pokazati da njena inverzna f-ja $h^{-1} = g$ zadovoljava 2 uvjeta iz definicije:

$g(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$

1) $g \in C^k$ (ako je $f \in C^k$)

2)
$$g' = \begin{array}{cl} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline f_{x_1} & \cdots & \cdots & f_{x_n} \end{pmatrix}_{(n+1) \times n} & \begin{array}{l} - \text{za } x_1 \\ - \text{za } x_2 \\ \\ \\ \\ \end{array} \end{array}$$

$\text{rang } g' = n$